# Linear Algebra & Random Processes

## Transformations and Expectation of Random Variables

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# Topics Covered & References

### Topics

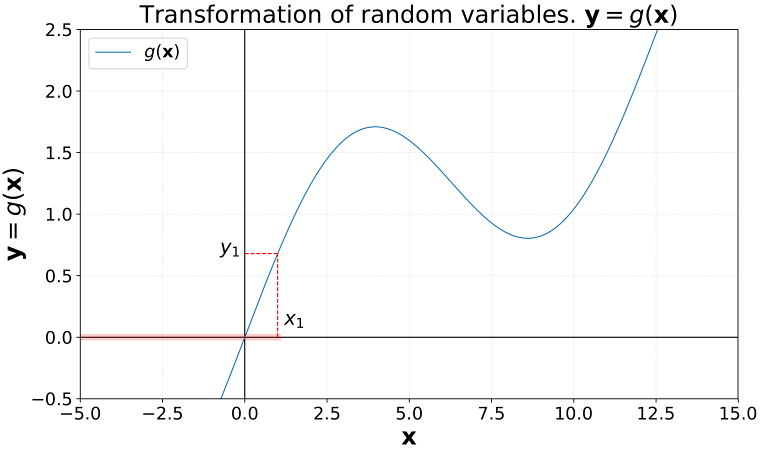
### References

# Transformation of random variables

- We are often interested in functions of random variables and their probability distributions.
- Consider a random variable $\mathbf{x} : S \mapsto \mathbb{R}$ with a know pdf or pmf. A function of the r.v. $\mathbf{x}$, represented as $g(\mathbf{x})$, is another r.v. $\mathbf{y} = g(\mathbf{x}) : S \mapsto \mathbb{R}$.
- For any given run of the random experiment with outomes $\zeta$, the value assumed by $\mathbf{y}$ can be obtained from the values assumed by $\mathbf{x}$, i.e $y = g(x) = g(\mathbf{x}(\zeta)) = \mathbf{y}(\zeta)$.
- Can we obtain the probability distribution of $\mathbf{y}$ using our knowledge of $f_{\mathbf{x}}(\cdot), F_{\mathbf{x}}(\cdot)$ and $g(\cdot)$?

$$F_{\mathbf{y}}(y) = P(\mathbf{y} \leq y) = P(g(\mathbf{x}) \leq y)$$

# Transformation of random variables



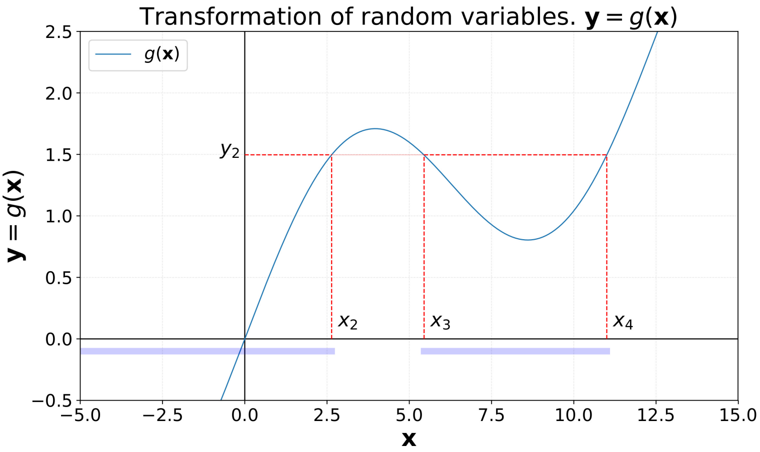

Let first consider $y_1$,

$$P(\mathbf{y} \leq y_1) = P(\mathbf{x} \in R_{y_1})$$
$$R_{y_1} = \{x \,|\, g(x) \leq y_1\}$$
$$= \{x \,|\, x \leq x_1\}$$

Thus, we have

$$F_{\mathbf{y}}(y_1) = F_{\mathbf{x}}(x_1)$$

# Transformation of random variables




What about $y_2$?

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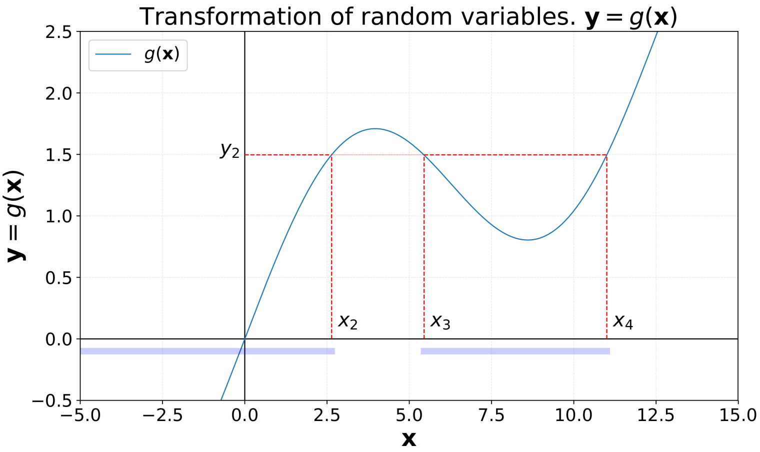


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$$P(\mathbf{y} \leq y_2) = P(\mathbf{x} \in R_{y_2})$$

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$$= \{x \,|\, x \leq x_2\} \cup$$

$$\{x \,|\, x_3 \leq x \leq x_4\}$$

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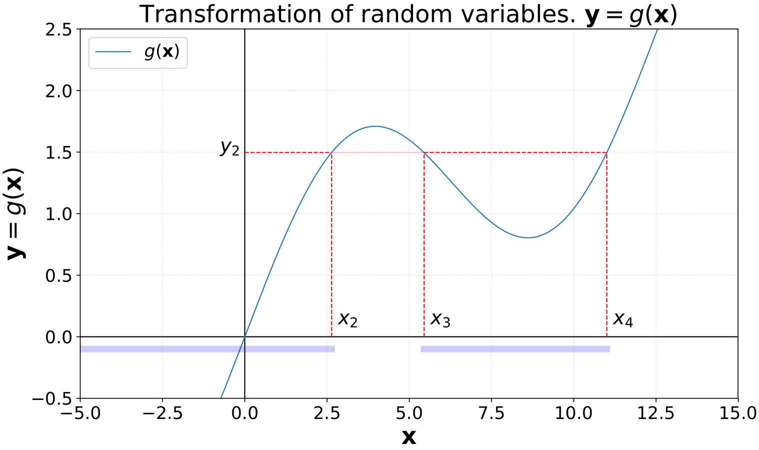


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$$F_{\mathbf{y}}(y_1) =$$

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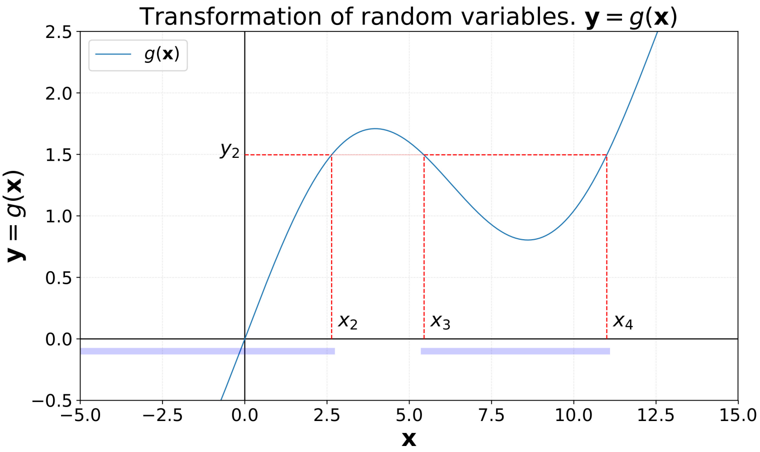


What about $y_2$?

$$P\left(\mathbf{y} \leq y_2\right) = P\left(\mathbf{x} \in R_{y_2}\right)$$

$$R_{y_2} = \left\{x \,|\, g\left(x\right) \leq y_2\right\}$$

$$= \left\{x \,|\, x \leq x_2\right\} \cup$$

$$\left\{x \,|\, x_3 \leq x \leq x_4\right\}$$

Thus, we have

$$F_{\mathbf{y}}\left(y_1\right) = F_{\mathbf{x}}\left(x_2\right) + F_{\mathbf{x}}\left(x_4\right)$$

$$- F_{\mathbf{x}}\left(x_3\right)$$

# Transformation of random variables

- Things are a litte easier when $g(\cdot)$ is a monotonic function. In this case, $g^{-1}(\cdot)$ exists, and $R_y = g^{-1}(\cdot)$. $\implies F_{\mathbf{y}}(y) = F_{\mathbf{x}}\left(g^{-1}(y)\right)$.

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  Let $\mathbf{x}$ by a uniformly distributed random variables between $[1, 3]$, and let $\mathbf{y} = \log(\mathbf{x})$.

  What is $F_{\mathbf{y}}(\cdot)$?

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- We can derive $f_{\mathbf{y}}(\cdot)$ from $F_{\mathbf{y}}(\cdot)$ by differentiation w.r.t. $y$. In the case of a monotonic $g(\cdot)$, we have

$$f_{\mathbf{y}}(\cdot) = f_{\mathbf{x}}\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|$$

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  Verify this for the previous examples.

# Expectation of Random Variables

- The expectation of a random variable is the **average or mean value of the r.v.**, which is formally defined as the following,

$$\mu_{\mathbf{x}} = \mathbb{E}\left(\mathbf{x}\right) \triangleq \begin{cases} \int_{-\infty}^{\infty} x \cdot f_{\mathbf{x}}\left(x\right) dx, & \mathbf{x} \text{ is continuous} \\ \sum_{x=-\infty}^{\infty} x \cdot f_{\mathbf{x}}\left(x\right), & \mathbf{x} \text{ is discrete} \end{cases}$$

  The expectation operation is a weighted sum of the different possible values assumed by $\mathbf{x}$, with the weights given by the pdf or pmf.

- The expectation operation can be defined for any general function of a r.v. $g\left(\mathbf{x}\right)$,

$$\mathbb{E}\left(g\left(\mathbf{x}\right)\right) = \int_{-\infty}^{\infty} g\left(x\right) \cdot f_{\mathbf{x}}\left(x\right) dx$$

# Expectation of Random Variables

- We can also calculate these with respect to conditional pdf and pmf.

$$\mathbb{E}\left(\mathbf{x} \,|\, A\right) = \int_{-\infty}^{\infty} x \cdot f_{\mathbf{x}}\left(x \,|\, A\right) dx$$

  This is the **conditional mean** of $\mathbf{x}$.

- Properties of $\mathbb{E}\left(\mathbf{x}\right)$. Let $a, b, c$ be scalar constants, and let $\mathbb{E}\left(g_1\left(\mathbf{x}\right)\right)$ and $\mathbb{E}\left(g_2\left(\mathbf{x}\right)\right)$ exist. Then,
  1. $\mathbb{E}\left(ag_1\left(\mathbf{x}\right) + bg_2\left(\mathbf{x}\right) + c\right) = a\mathbb{E}\left(g_1\left(\mathbf{x}\right)\right) + b\mathbb{E}\left(g_2\left(\mathbf{x}\right)\right) + c$
  2. $g_1\left(\mathbf{x}\right) \geq 0 \implies \mathbb{E}\left(g_1\left(\mathbf{x}\right)\right) \geq 0$
  3. $g_1\left(\mathbf{x}\right) \geq g_2 ctx \implies \mathbb{E}\left(g_1\left(\mathbf{x}\right)\right) \geq \mathbb{E}\left(g_2\left(\mathbf{x}\right)\right)$
  4. $a \leq g_1\left(\mathbf{x}\right) \leq b \implies a \leq \mathbb{E}\left(g_1\left(\mathbf{x}\right)\right) \leq b$

# Expectation of Random Variables

- We can measure the spread of r.v. around its mean through the variance $\sigma_{\mathbf{x}}^2$,

$$\sigma_{\mathbf{x}}^2 = \mathbb{E}\left(\left(\mathbf{x} - \mu_{\mathbf{x}}\right)^2\right) = \int_{-\infty}^{\infty} \left(x - \mu_{\mathbf{x}}\right)^2 \cdot f_{\mathbf{x}}\left(x\right) dx = \mathbb{E}\left(\mathbf{x}^2\right) - \mu_{\mathbf{x}}^2$$

- Moments of $\mathbf{x}$.

$$m_n = \int_{-\infty}^{\infty} x^n \cdot f_{\mathbf{x}}\left(x\right) dx$$

- Central moments of $\mathbf{x}$.

$$c_n = \int_{-\infty}^{\infty} \left(x - \mu_{\mathbf{x}}\right)^n \cdot f_{\mathbf{x}}\left(x\right) dx$$

# Moment Generating Function of Random Variables

- We define the moment generating function of a r.v. $\mathbf{x}$ by the following,

$$\Phi_{\mathbf{x}}(s) \triangleq \mathbb{E}\left(e^{sx}\right) = \int_{-\infty}^{\infty} f_{\mathbf{x}}(x)\, e^{sx} dx$$

  $\mathbb{E}\left(e^{j\omega x}\right)$ is called the characteristic function of $\mathbf{x}$.

$$\Phi_{\mathbf{x}}(z) \triangleq \mathbb{E}\left(z^{x}\right) = \sum_{x=-\infty}^{\infty} f_{\mathbf{x}}(x)\, z^{x}$$

  $\mathbb{E}\left(e^{j\omega x}\right)$ is called the characteristic function of $\mathbf{x}$.

- The moments of $\mathbf{x}$ can be obtained from $\Phi_{\mathbf{x}}(s)$. $m_n = \Phi_{\mathbf{x}}^{(n)}(0)$.