Linear Systems

Matrices

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References

- S Boyd, Applied Linear Algebra: Chapters 6, 7, 8, 10 and 11.
- G Strang, Linear Algebra: Chapters 1 and 2.

Matrices

- **Matrices** are rectangular array of numbers. $\begin{bmatrix} 1.1 & -24 & \sqrt{2} \\ 0 & 1.12 & -5.24 \end{bmatrix}$

$$\begin{array}{ccc} & m \text{ columns} \longrightarrow & \\ n \text{ rows} \downarrow & \begin{bmatrix} \square & \square & \square & \cdots & \square \\ \square & \square & \square & \cdots & \square \\ \square & a_{32} & \square & \cdots & \square \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \square & \square & \square & \cdots & \square \end{bmatrix} & \longrightarrow 3^{rd} \text{ row} \\ & 2^{nd} \text{ column} & \end{array}$$

- Consider a matrix A with n rows and m columns. $\begin{cases} \textbf{Tall/Skinny} & n > m \\ \textbf{Square} & n = m \\ \textbf{Wide/Fat} & n < m \end{cases}$

Matrices

- n-vectors can be interpreted as $n \times 1$ matrices. These are called *column vectors*.
- A matrix with only one row is called a *row vector*, which can be referred to as n-row-vector. $\mathbf{x} = \begin{bmatrix} 1.45 & -3.1 & 12.4 \end{bmatrix}$
- **Block matrices & Submatrices**: $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$. What are the dimensions of the different matrices?
- Matrices are also compact way to give a set of indexed column n-vectors, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \ldots \mathbf{x}_m$.
$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \ldots & \mathbf{x}_m \end{bmatrix}$$
- **Zero matrix**$= \mathbf{0}_{n \times m} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}$

Matrices

- **Identity matrix** is a square $n \times n$ matrix with all zero elements, except the diagonals where all elements are 1.

$$\mathbf{I}_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}$$

- **Diagonal matrices** is a square matrix with non-zero elements on its diagonal.

$$\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & -11 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 9.3 \end{bmatrix} = \textbf{diag}\left(0.4, -11, 21, 9.3\right)$$

- **Triangular matrices**: Are square matrices. *Upper triangular* $a_{ij} = 0, \forall i > j$; *Lower triangular* $a_{ij} = 0, \forall i < j$.

Matrix operations

- **Transpose** switches the rows and columns of a matrix. $\mathbf{A}$ is a $n \times m$ matrix, then its transpose is represented by $\mathbf{A}^T$, which is a $m \times n$ matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \implies \mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

  Transpose converts between column and row vectors.

  What is the transpose of a block matrix? $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$

- **Matrix addition** can only be carried out with matrices of same size. Like vectors we perform element wise addition.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Matrix operations

- **Properties of matrix addition**:
  - *Commutative*: $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
  - *Associative*: $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
  - *Addition with zero matrix*: $\mathbf{A}+\mathbf{0}=\mathbf{0}+\mathbf{A}=\mathbf{A}$
  - *Transpose of sum*: $(\mathbf{A}+\mathbf{B})^T=\mathbf{A}^T+\mathbf{B}^T$
- **Scalar multiplication** Each element of the matrix gets multiplied by the scalar.

$$\alpha\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}\alpha a_{11} & \alpha a_{12}\\ \alpha a_{21} & \alpha a_{22}\end{bmatrix}$$

- We will mostly only deal with matrices with real entries. Such matrices are elements of the set $\mathbb{R}^{n\times m}$.
- Given the aforementioned matrix operations and their properties, is $\mathbb{R}^{n\times m}$ a vector space?

Matrix multiplication

- It is possible to *multiply* two matrices $\mathbf{A} \in \mathbb{R}^{n \times p}$ and $\mathbf{B} \in \mathbb{R}^{p \times m}$ through *matrix multiplication* procedure.
- There is a product matrix $\mathbf{C} \coloneqq \mathbf{AB} \in \mathbb{R}^{n \times m}$, if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

$$c_{ij} \coloneqq \sum_{k=1}^{p} a_{ik} b_{kj} \quad \forall i \in \{1, \ldots n\} \;\; \& \;\; j \in \{1 \ldots m\}$$

- *Inner product* is a special case of matrix multiplication between a *row vector* and a *column vector*.

$$\mathbf{x}^T \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$$

Matrix multiplication

- Consider a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ and a m-vector $\mathbf{x} \in \mathbb{R}^m$. We can multiply $\mathbf{A}$ and $\mathbf{x}$ to obtain $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^n$.

$$\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{a}}_1^T \\ \tilde{\mathbf{a}}_2^T \\ \vdots \\ \tilde{\mathbf{a}}_n^T \end{bmatrix} x = \begin{bmatrix} \tilde{\mathbf{a}}_1^T x \\ \tilde{\mathbf{a}}_2^T x \\ \vdots \\ \tilde{\mathbf{a}}_n^T x \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i} x_i \\ \sum_{i=1}^m a_{2i} x_i \\ \vdots \\ \sum_{i=1}^m a_{ni} x_i \end{bmatrix}$$

$$\mathbf{y} = \sum_{i=1}^{m} x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \dots + x_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix}$$

- Multiplying a matrix $\mathbf{A}$ by a column vector $\mathbf{x}$ produces a linear combination of the columns of matrix $\mathbf{A}$. The column mixture is provided by $\mathbf{x}$.

Matrix multiplication

- We see a similar process in play when we multiply a row vector $\mathbf{x}^T \in \mathbb{R}^n$ by a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$.

$$\mathbf{y} = \mathbf{x}^T \mathbf{A} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} = \mathbf{x}^T \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_m \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{x}^T \mathbf{a}_1 & \mathbf{x}^T \mathbf{a}_2 & \dots & \mathbf{x}^T \mathbf{a}_m \end{bmatrix} = \sum_{i=1}^{n} x_i \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{im} \end{bmatrix}$$

- Multiplying a row vector $\mathbf{x}$ by a matrix $\mathbf{A}$ produces a linear combination of the row of matrix $\mathbf{A}$. The row mixture is provided by $\mathbf{x}$.

Matrix multiplication

- Multiplying two matrices $\mathbf{A} \in \mathbb{R}^{n \times p}$ and $\mathbf{B} \in \mathbb{R}^{p \times m}$, we have $\mathbf{C} \in \mathbb{R}^{n \times m}$,

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{n2} & \dots & c_{nm} \end{bmatrix}$$

- **Inner product interpretation**: $c_{ij} = \tilde{\mathbf{a}}_i^T \mathbf{b}_j,\ i \in \{1 \dots n\},\ j \in \{1 \dots m\}$
- **Column interpretation**: $\mathbf{C} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \dots & \mathbf{A}\mathbf{b}_m \end{bmatrix}$
- **Row interpretation** : $\mathbf{C} = \begin{bmatrix} \tilde{\mathbf{a}}_1^T \\ \tilde{\mathbf{a}}_2^T \\ \vdots \\ \tilde{\mathbf{a}}_n^T \end{bmatrix} \mathbf{B} = \begin{bmatrix} \tilde{\mathbf{a}}_1^T \mathbf{B} \\ \tilde{\mathbf{a}}_2^T \mathbf{B} \\ \vdots \\ \tilde{\mathbf{a}}_n^T \mathbf{B} \end{bmatrix}$

Matrix multiplication

- **Outer product interpretation** Consider two n-vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The *outer product* is defined as,

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 & \dots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \dots & x_1y_n \\ x_2y_1 & x_2y_2 & x_2y_3 & \dots & x_2y_n \\ x_3y_1 & x_3y_2 & x_3y_3 & \dots & x_3y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & x_ny_3 & \dots & x_ny_n \end{bmatrix}$$

- We can represent the product between two matrices as the sum of outer products between the columns and $\mathbf{A}$ and rows of $\mathbf{B}$.

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \dots & \mathbf{a}_p \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{b}}_1^T \\ \tilde{\mathbf{b}}_2^T \\ \tilde{\mathbf{b}}_3^T \\ \vdots \\ \tilde{\mathbf{b}}_p^T \end{bmatrix} = \sum_{i=1}^{p} a_i \tilde{b}_i^T$$

Properties of matrix multiplication

- **Not commutative**: $\mathbf{AB} \neq \mathbf{BA}$
  The product of two matrices might not alwasys be defined. When it is defined, $\mathbf{AB}$ and $\mathbf{BA}$ need not match.
- **Distributive**: $\mathbf{A}\left(\mathbf{B} + \mathbf{C}\right) = \mathbf{AB} + \mathbf{BC}$ and $\left(\mathbf{A} + \mathbf{B}\right)\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
- **Associative**: $\mathbf{A}\left(\mathbf{BC}\right) = \left(\mathbf{AB}\right)\mathbf{C}$
- **Transpose**: $\left(\mathbf{AB}\right)^T = \mathbf{B}^T\mathbf{A}^T$
- **Scalar product**: $\alpha\left(\mathbf{AB}\right) = \left(\alpha\mathbf{A}\right)\mathbf{B} = \mathbf{A}\left(\alpha\mathbf{B}\right)$

Linear equations

- Matrices present a compact way to represent a set of linear equations. Consider the following,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 \ldots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 \ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \longrightarrow \mathbf{A}\mathbf{x} = \mathbf{b}, \;\; \mathbf{A} \in \mathbb{R}^{m \times n}, \;\; \mathbf{x} \in \mathbb{R}^n, \;\; \mathbf{b} \in \mathbb{R}^m$$

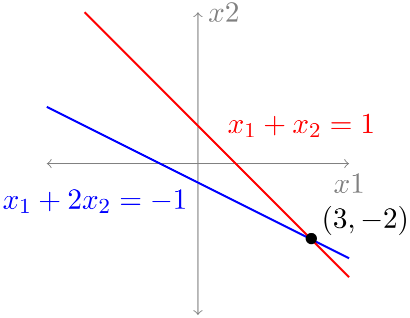
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Geometry of linear equations

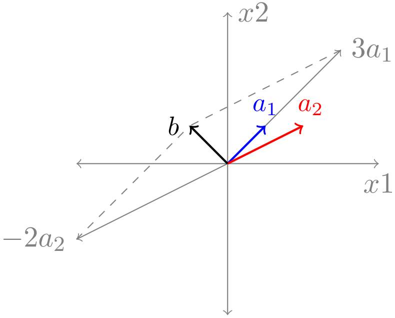
$$\left.\begin{aligned} x_1 + 2x_2 &= -1 \\ x_1 + x_2 &= 1 \end{aligned}\right\} \longrightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Two ways to view this: **row view** and the **column view**.

Traditional (row) view



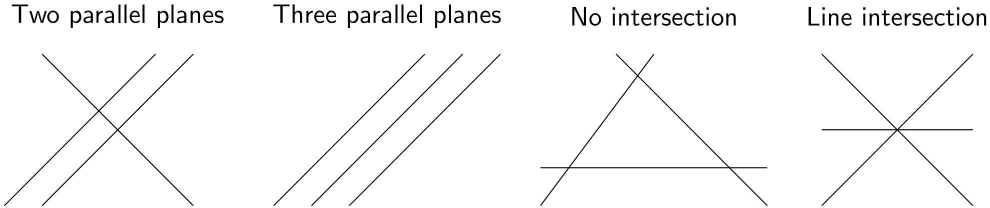
Column view



Solving linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \;\; \mathbf{A} \in \mathbb{R}^{m \times n}, \;\; \mathbf{x} \in \mathbb{R}^{n}, \;\; \mathbf{b} \in \mathbb{R}^{m}$$

- **Three possible situations:** No solution, Infinitely many solutions, or Unique Solution.
- When do have infinitely many or no solutions? In $\mathbb{R}^3$, we can visualize the different situations.



Solving linear equations: Gaussian Elimination

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n &= b_1 : E_1 \\
a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n &= b_2 : E_2 \\
a_{31}x_1 + a_{32}x_2 \ldots + a_{3n}x_n &= b_3 : E_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 \ldots + a_{mn}x_n &= b_m : E_m
\end{aligned}$$

- Gaussian elimination is a systematic way of simplifying the above equations to an equivalent system that can be easily solved.
- Three simple operations are repeatedly performed:
  - Interchanging of equations E_i and E_j.
  - Replacing equation E_i by αE_i, $\alpha \neq 0$.
  - Replacing equation E_j by $E_j + \alpha E_i$, $\alpha \neq 0$.
- These three operations do not change the solution of the given linear system.

Solving linear equations: Gaussian Elimination

Augmented matrix: $$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- We can work with the augmented matrix instead of the equations.
- Gaussian elimination is carried out on the entire matrix.
- The matrix is simplified to a point, from where one can easily:
  - find out the nature of the solutions for the system of equations; and
  - find the solution (with a bit of extra work), if they exist.

Solving linear equations: Gaussian Elimination

$$\left.\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 4 \\ -2x_1 & - & 4x_2 & + & x_3 & = & -3 \end{array}\right\} \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 2 & 3 & 4 & 4 \\ -2 & -4 & 1 & -3 \end{array}\right]$$

Gaussian Elimination

$$\left[\begin{array}{rrr|r} \underline{1} & 2 & -1 & 1 \\ 2 & 3 & 4 & 4 \\ -2 & -4 & 1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} \underline{1} & 2 & -1 & 1 \\ 0 & -1 & 6 & 2 \\ -2 & -4 & 1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} \underline{1} & 2 & -1 & 1 \\ 0 & \underline{-1} & 6 & 2 \\ 0 & 0 & \underline{-1} & -1 \end{array}\right]$$

Now, we can perform **back substitution** on this triangularized system of linear equations,

$$x_3 = 1; \;\; x_2 = 4; \;\; x_1 = -6$$

We can continue the simplification process through the **Gauss-Jordan** method.

Solving linear equations: Gauss-Jordan Method

Continue the elimination upwards until all elements, except the ones in the main diagonal, are zero.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 6 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \implies x_1 = -6; \;\; x_2 = 4; \;\; x_3 = 1;$$

Everything worked out well without any problems. What can go wrong here?

Try solving the these systems, $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 4 & 4 \\ -2 & -4 & 2 & -3 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 4 & 4 \\ -2 & -4 & 2 & -2 \end{array}\right]$

What is the difference between these two systems?

Solving linear equations: Rectangular systems and Row Echelon Form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n &= b_1 : E_1 \\ a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n &= b_2 : E_2 \\ a_{31}x_1 + a_{32}x_2 \ldots + a_{3n}x_n &= b_3 : E_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 \ldots + a_{mn}x_n &= b_m : E_m \end{aligned} \longrightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Consider the following example,

$$\left[\begin{array}{ccccc|c} \underline{1} & -2 & 1 & 0 & 1 & 1 \\ 2 & -4 & 1 & -1 & -2 & 2 \\ -1 & 2 & 1 & 1 & 2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \underline{1} & -2 & 1 & 0 & 1 & 1 \\ 0 & 0 & \underline{-1} & -1 & -4 & 0 \\ 0 & 0 & 2 & 1 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \underline{1} & -2 & 1 & 0 & 1 & 1 \\ 0 & 0 & \underline{-1} & -1 & -4 & 0 \\ 0 & 0 & 0 & \underline{-1} & -5 & 0 \end{array}\right]$$

Solving linear equations: Rectangular systems and Row Echelon Form

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \longrightarrow \begin{bmatrix} \underline{*} & * & * & * & * & * & * \\ 0 & 0 & \underline{*} & * & * & * & * \\ 0 & 0 & 0 & \underline{*} & * & * & * \\ 0 & 0 & 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Things to notice about the echelon form:

- If a particular row consists entirely of zeros, then all rows below that row also contain entirely of zeros.
- If the first non-zero entry in the i^{th} row occurs in the j^{th} position, then all elements below the i^{th} row are zero from columns 1 to j.

Columns containing pivot are called the **basic columns**.

Rank of a matrix $\mathbf{A}$ is defined at the number of basic columns in the row echelon form of the matrix $\mathbf{A}$.

Solving linear equations: Reduced Row Echelon Form

$$\begin{bmatrix} \underline{*} & * & * & * & * & * & * \\ 0 & 0 & \underline{*} & * & * & * & * \\ 0 & 0 & 0 & \underline{*} & * & * & * \\ 0 & 0 & 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} \underline{1} & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & * & 0 \\ 0 & 0 & 0 & \underline{1} & 0 & * & 0 \\ 0 & 0 & 0 & 0 & \underline{1} & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} \underline{1} & -2 & 1 & 0 & 1 & 1 \\ 0 & 0 & \underline{-1} & -1 & -4 & 0 \\ 0 & 0 & 0 & \underline{-1} & -5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \underline{1} & -2 & 0 & -1 & -3 & 1 \\ 0 & 0 & \underline{1} & 1 & 4 & 0 \\ 0 & 0 & 0 & \underline{-1} & -5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \underline{1} & -2 & 0 & 0 & 2 & 1 \\ 0 & 0 & \underline{1} & 0 & -1 & 0 \\ 0 & 0 & 0 & \underline{1} & 5 & 0 \end{array}\right]$$

- All non-basic columns can be represented as a linear combination of the basic columns.
- A non-basic column is a linear combination of only the columns before it.
- Scaling factors for each basic columns is determined by the corresponding elements of the non-basic columns.

The reduced row echelon form reveals structure in the original matrix A.

Solving linear equations: Homogenous Systems

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 \ldots + a_{3n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 \ldots + a_{mn}x_n &= 0 \end{aligned} \longrightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ a_{31} & a_{32} & \cdots & a_{3n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

Consider the following case,

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 1 & 0 & 1 & 0 \\ 2 & -4 & 1 & -1 & -2 & 0 \\ -1 & 2 & 1 & 1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + 2x_5 &= 0 \\ x_3 - x_5 &= 0 \\ x_4 + 5x_5 &= 0 \end{aligned} \longrightarrow \begin{aligned} x_1 &= 2x_2 - 2x_5 \\ x_3 &= x_5 \\ x_4 &= -5x_5 \end{aligned} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 2x_5 \\ x_2 \\ x_5 \\ 5x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

Solving linear equations: Homogenous Systems

- $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 1 \\ -5 \\ 1 \end{bmatrix}$ represents the general solution of the system of equations.
- In general, any system $[\mathbf{A} \mid \mathbf{0}]$ with $rank\,(\mathbf{A}) = r$ and $r < n$ has the general solution of the form,

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \ldots + x_{f_{n-r}}\mathbf{h}_{n-r}$$

  where, the variables $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are called the **free variables**.
- Free variables are the one corresponding to the non-basic columns; the variable variables corresponding to the basic columns are the **basic variables**.
- When does a homogenous system have a unique solution solution? $\longrightarrow rank\,(\mathbf{A}) = n$.

Solving linear equations: Non-homogenous Systems

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 \ldots + a_{3n}x_n &= b_3 \quad \longrightarrow [\mathbf{A} \mid \mathbf{b}] \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Consider the following case,

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 1 & 1 \\ 2 & -4 & 1 & -1 & -2 & 2 \\ -1 & 2 & 1 & 1 & 2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + 2x_5 &= 1 & x_1 &= 1 + 2x_2 - 2x_5 \\ x_3 - x_5 &= 0 \longrightarrow & x_3 &= x_5 \\ x_4 + 5x_5 &= 0 & x_4 &= -5x_5 \end{aligned} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 + 2x_2 - 2x_5 \\ x_2 \\ x_5 \\ 5x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

The general solution of a non-homogenous system is sum of the particular solution and the general solution of the associated homogenous system.

Solving linear equations: Non-homogenous Systems

- The general solution for $[A \mid 0]$ with $rank\,(A) = r$,

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \ldots + x_{f_{n-r}}\mathbf{h}_{n-r}$$

  where, $\mathbf{p}$ is the particular solution and $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables.
- When do we have a unique solution to this system? $\longrightarrow rank\,(\mathbf{A}) = n$.
- What about the case when there are no solutions? When does that happen? $\longrightarrow$ *When the system is not consistent.*

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} 1 & * & 0 & 0 & 0 & * & c_1 \\ 0 & 0 & 1 & 0 & 0 & * & c_2 \\ 0 & 0 & 0 & 1 & 0 & * & c_3 \\ 0 & 0 & 0 & 0 & 1 & * & c_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & c_m \end{array}\right]$$

  There is a problem when $c_m \neq 0$
- The augmented matrix $[\mathbf{A} \mid \mathbf{b}]$ has the same number of basic columns as $\mathbf{A}$.
- $[\mathbf{A} \mid \mathbf{b}] \rightarrow [\mathbf{E} \mid \mathbf{c}]$: $\mathbf{c}$ is a non-basic column.
- $rank\,(\mathbf{A}) = rank\,([\mathbf{A} \mid \mathbf{b}])$

$\mathbf{LU}$ Factorization of a Matrix

- A major theme of matrix algebra is to decompose matrices into simpler components that provide insights into the nature of the matrix.
- A full rank square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ can be decomposed into the product of a lower triangular and an upper triangular matrix.
- Matrices associated with the three elementary operations:

Inter-changing rows 2 and 4	**Scaling row 2**	**Adding a multiple of row 2 to row 3**
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

LU Factorization of a Matrix

- Consider the case: $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -1 \end{bmatrix} = \mathbf{LU}$
- $\mathbf{LU}$ factorization can be done only when no zero pivot is encountered during the Guassian elimination process.
- $\mathbf{Ax} = \mathbf{b}$ becomes $\mathbf{LUx} = \mathbf{b}$: This is decomposed into two triangular systems, $\mathbf{Ux} = \mathbf{y}$, $\mathbf{Ly} = \mathbf{b}$. First solve $\mathbf{Ly} = \mathbf{b}$ and then solve $\mathbf{Ux} = \mathbf{y}$
- Properties:
  - Diagonal elements of $\mathbf{L}$ are 1, and $\mathbf{U}$ are not equal to zero.
  - $\mathbf{U}$ is the final result of Gaussian elimination, and $\mathbf{L}$ is the matrix that reverses this process.
  - Element l_{ij} of $\mathbf{L}$ is the multiple of row j used to eliminate the a_{ij} element of $\mathbf{A}$.
- Uses of the $\mathbf{LU}$ factorization:
  - Solving $\mathbf{Ax} = \mathbf{b}_i$ for several $\mathbf{b}_i$s. $\mathbf{LU}$ need to be calculated only once.
  - Factorization requires no extra space.

$\mathbf{PA} = \mathbf{LU}$ Factorization of a Matrix

- Consider the case: $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \mathbf{LU}$
- It turns out the second pivot become zero after the first elimination step, so $\mathbf{LU}$ factorization cannot be done on $\mathbf{A}$.
- The following however fixes this issue,

$$\mathbf{PA} = \mathbf{LU}$$

  where, $\mathbf{P}$ is the permutation matrix, which is the elementary matrix for row exchanges.
- In the current example, the following allows matrix factorization.

$$\mathbf{PA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{LU}$$

Linear transformations

- We had earlier seen linear functions of the form $f : \mathbb{R}^n \mapsto \mathbb{R}$, which could be expressed as,
$$y = f(\mathbf{x}) = \mathbf{w}^T\mathbf{x}; \;\; \mathbf{w}, \mathbf{x} \in \mathbb{R}^n, \; y \in \mathbb{R}$$
- A general version is when the range of the function is not in $\mathbb{R}$ but in $\mathbb{R}^m$:
$$\mathbf{y} = f(\mathbf{x}); \;\; \mathbf{x} \in \mathbb{R}^n, \; \mathbf{y} \in \mathbb{R}^m$$
- Such a function has a natural representation of the form $\mathbf{y} = \mathbf{A}\mathbf{x}, \;\; \mathbf{A} \in \mathbb{R}^{m \times n}$.
- Any linear transformation can be expressed as $\mathbf{y} = \mathbf{A}\mathbf{x}$.
- Matrices can be thought of as representing a particular linear transformation.

Another look at matrix multiplication

Why does matrix multiplication have this strange definition?

Consider the following two functions,

$$\mathbf{y} = f\left(\mathbf{x}\right) = \mathbf{A}\mathbf{x} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{v} = g\left(\mathbf{u}\right) = \mathbf{B}\mathbf{u} \longrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{z} = h\left(\mathbf{u}\right) = f\left(g\left(\mathbf{u}\right)\right) = f\left(\begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix}\right) &= \begin{bmatrix} a\alpha u_1 + a\beta u_2 + b\gamma u_1 + b\delta u_2 \\ c\alpha u_1 + c\beta u_2 + d\gamma u_1 + d\delta u_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(a\alpha + b\gamma\right) u_1 + \left(a\beta + b\delta\right) u_2 \\ \left(c\alpha + d\gamma\right) u_1 + \left(c\beta + d\delta\right) u_2 \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

$$\mathbf{z} = \mathbf{A}\left(\mathbf{B}\mathbf{u}\right) = \left(\mathbf{A}\mathbf{B}\right)\mathbf{u} \implies \mathbf{A}\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

This definition of matrix multiplication is the most natural for dealing with composition of linear transformations.

Four Fundamental Subspaces

- $C(\mathbf{A})$: **Column Space of $\mathbf{A}$** – the span of the columns of $\mathbf{A}$.

$$C(\mathbf{A}) = \{\mathbf{A}\mathbf{x} \,|\, \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

- $N(\mathbf{A})$: **Nullspace of $\mathbf{A}$** – the set of all $\mathbf{x} \in \mathbb{R}^n$ that are mapped to zero by $\mathbf{A}$.

$$N(\mathbf{A}) = \{\mathbf{x} \,|\, \mathbf{A}\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

- $C(\mathbf{A}^T)$: **Row Space of $\mathbf{A}$** – the span of the rows of $\mathbf{A}$.

$$C(\mathbf{A}^T) = \{\mathbf{A}^T\mathbf{y} \,|\, \mathbf{y} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

- $N(\mathbf{A}^T)$: **Nullspace of $\mathbf{A}^T$** – the set of all $\mathbf{y} \in \mathbb{R}^m$ that are mapped to zero by $\mathbf{A}^T$.

$$N(\mathbf{A}^T) = \{\mathbf{y} \,|\, \mathbf{A}^T\mathbf{y} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

This is also called the **left nullspace** of $\mathbf{A}$.

Linear Independence

- Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}, \;\; \mathbf{v}_i \in \mathbb{R}^m$, how can we determine if this set is linear independent?
- We need to verify, $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0$

$$\left.\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{Va} = \mathbf{0} \right\} N(\mathbf{V}) = \{\mathbf{0}\}, \;\; rank(\mathbf{V}) = n$$

- This is also equivalent to saying that when the $rank(\mathbf{A}) = n \implies$ the columns of $\mathbf{A}$ form an independent set of vectors.
- When do the rows of A form an independent set?
- What about both rows and columns? When does that happen?

Examples

- $$\underbrace{\begin{bmatrix} 2 & -1 & 0 & 1 \\ 4 & 3 & 1 & -1 \\ 6 & -2 & 5 & 1 \\ 2 & 3 & 0 & -1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 6 \end{bmatrix}}_{\mathbf{b}}$$
- Now solve the above for a different $\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.
- $$\begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ 6 & -2 & 5 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 13 \\ -17 \\ 14 \end{bmatrix}$$
- Now solve the above for a different $\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.
- $$\begin{bmatrix} 2 & -1 & -2 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
- Now solve the above for a different $\mathbf{b} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.
- Reduce this to the row echelon form:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Dimension and basis of the four fundamental subspaces

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 1 \\ -1 & 2 & 2 \end{bmatrix}; \quad \mathbf{EA} = \mathbf{R}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}}_{\mathbf{E}} \mathbf{A} = \underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{R}$$

Pivot cols of $\mathbf{A}$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ $N(\mathbf{A})$: $x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

We can restructure $\mathbf{EA} = \mathbf{R} \rightarrow \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{bmatrix} \mathbf{A} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix}$

- **Column space $C(\mathbf{A})$**
  - $\dim C(\mathbf{A}) = rank(\mathbf{A}) = r$
  - Basis of $C(\mathbf{A})$ = Pivot colums of $\mathbf{A}$.
- **Nullspace $N(\mathbf{A})$**
  - $\dim N(\mathbf{A}) = n - r$
  - Basis of $N(\mathbf{A}) = \{\mathbf{h}_1, \mathbf{h}_2 \ldots \mathbf{h}_{n-r}\}$.
- **Row space $C(\mathbf{A}^T)$**
  - $\dim C(\mathbf{A}^T) = rank(\mathbf{A}^T) = rank(\mathbf{A}) = r$
  - Basis of $C(\mathbf{A}^T)$ = Columns of $\mathbf{R}_1^T$.
- **Left Nullspace $N(\mathbf{A}^T)$**
  - $\dim N(\mathbf{A}^T) = m - r$
  - Basis of $N(\mathbf{A}^T)$ = Colums of $\mathbf{E}_2^T$

Matrix Inverse

- Consider the square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. $\mathbf{B} \in \mathbb{R}^{n \times n}$ is the inverse of $\mathbf{A}$, if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, and $\mathbf{B}$ is represented as $\mathbf{A}^{-1}$.
- Not all matrices have inverses. A matrix with an inverse is called **non-singular**, otherwise it is called **singular**.
- For a non-singular matrix $\mathbf{A}$, $\mathbf{A}^{-1}$ is unique. $\mathbf{A}^{-1}$ is both the left and right inverse.
- A matrix $\mathbf{A}$ has an inverse, if and only if $\mathbf{A}$ is full rank, i.e. $rank\left(\mathbf{A}\right) = n$
- The inverse of a non-signular matrix can be determined through Gauss-Jordan method. $\left[\mathbf{A} | \mathbf{I}\right] \xrightarrow{\text{Gauss-Jordan}} \left[\mathbf{I} | \mathbf{A}^{-1}\right]$. Lets try: $\begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$
- $\mathbf{Ax} = \mathbf{b}$ can be solved as follows, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. *It is never solved like this in practice.*
- Inverse of product of matrices, $\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$ and $\left(\mathbf{A}^{-1}\right)^{T} = \left(\mathbf{A}^{T}\right)^{-1}$