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Linear Systems Positive Definiteness and Matrix Norm

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References

► G Strang, Linear Algebra: Chapters 6 and 7.

- We know that $\mathbf{x}^T \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$.
- What can we say about $\mathbf{x}^T \mathbf{A} \mathbf{x}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$?
 - Is it positive for all $\mathbf{x} \neq \mathbf{0}$?
 - Can it be zero for some $\mathbf{x} \neq \mathbf{0}$?
 - Can it be negative some $\mathbf{x} \neq \mathbf{0}$?
- Any matrix A for which x^TAx > 0, for all x ≠ 0 is called a *positive definite matrix*.
- Positive definite matrices are very useful and are commonly encountered in practice: optimization, mechanics (mass matrix, stiffness matrix), stability analysis, co-variance matrices etc.

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Are the following matrices positive definite: $\begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 5 & -1 \\ 14 & 11 \end{bmatrix}$

Positive Definite Matrix

- The idea of positive definiteness is intimately related to the problem of minimization of a function.
- ▶ Consider the following function of a single variable f(x). This function reaches a minimum at x = 0, when $\frac{df(x)}{dx}|_{x=0} = 0$ and $\frac{d^2f(x)}{dx^2}|_{x=0} > 0$. E.g.,

$$f(x) = 3x^2 \rightarrow \frac{df(x)}{dx}\Big|_{x=0} = 0, \ \frac{d^2f(x)}{dx^2}\Big|_{x=0} = 3 > 0$$

▶ What about $f(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$? We can extend the previous idea using partial derivatives.

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = 0, \ \frac{\partial f(x_1, x_2)}{\partial x_2} = 0, \ \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} > 0, \ \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} > 0$$

Is this enough? $\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}$ must also be taken into account.

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▶ We can rearrange $ax_1^2 + 2bx_1x_2 + cx_2^2$ in the following manner,

$$f(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 = a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(c - \frac{b^2}{a}\right)x_2^2$$

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$$\frac{\partial^2 f\left(x_1, x_2\right)}{\partial x_1^2} > 0 \quad \text{and} \quad \frac{\partial^2 f\left(x_1, x_2\right)}{\partial x_1^2} \frac{\partial^2 f\left(x_1, x_2\right)}{\partial x_2^2} > \left(\frac{\partial^2 f\left(x_1, x_2\right)}{\partial x_1 \partial x_2}\right)^2$$

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Verfy this on the following functions: 1) $x_1^2 + x_1x_2 + x_2^2$, 2) $x_1^2 + 2x_1x_2 + x_2^2$, 3) $x_1^2 + 3x_1x_2 + x_2^2$

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Positive Definite Matrix

 $f(\bullet)$ can be expressed as $\mathbf{x}^T \mathbf{A} \mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, when \mathbf{A} is positive definite.

 $\mathbf{x}^T \mathbf{A} \mathbf{x}$ is called a *quadratic form*. For a symmetric matrix \mathbf{A} ,

$$\mathbf{x}^{T}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_{1} & x_{2} & \dots & x_{n} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{1} \\ x_{2} \\ \dots \\ x_{n} \end{bmatrix} = \sum_{i=1}^{n} a_{ii}x_{i}^{2} + 2\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_{ij}x_{i}x_{j}$$

In general, $\mathbf{A} \in \mathbb{R}^{n imes n}$ is positive definite, if:

- ► The eigenvalues of A are all positive.
- ▶ The pivots (without row exchange) are all positive.

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In general, $\mathbf{A} \in \mathbb{R}^{n imes n}$ is positive definite, if:

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Show that any A is positive definite if the symmetric matrix $A + A^T$ is positive definite. Note: This should explain why we have only been talking about symmetric matrices.

Matrix Norm

- Since matrices also form vector spaces, we can talk about norms of matrices, which extent the idea of sizes and distances to spaces of matrices.
- If we think of matrices a set of mn scalars, then we can use the same approach as vectors,

$$\|\mathbf{A}\|_{F} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^{2}\right)^{\frac{1}{2}}$$

This is called the Frobinius norm.

Matrix Norm

- There are other norms defined for matrices that are very useful from the point-of-view of linear transformation.
- These are called *induced matrix norms*, that looks at how matrices map vectors from the domain to range spaces.
- Let $\mathbf{A} \in \mathbb{R}^{m imes n} : \mathbb{R}^n \to \mathbb{R}^m$, and $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$

$$\|\mathbf{y}\| = \|\mathbf{A}\mathbf{x}\| \le C \|\mathbf{x}\|, \ \forall \mathbf{x} \in \mathbb{R}^n, \ C \ge 0$$

C is the maximum factor by which \mathbf{A} amplifies the vector \mathbf{x} .

The induced norm of a matrix is defined as,

$$\|\mathbf{A}\|_p = \max_{\mathbf{x}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p=1} \|\mathbf{A}\mathbf{x}\|_p$$

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Matrix Norm

Consider a matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{a}}_1^T \\ \tilde{\mathbf{a}}_2^T \end{bmatrix}.$$

$$\|\mathbf{A}\|_{1} = \max_{i} \|\mathbf{a}_{i}\|_{1} \qquad \|\mathbf{A}\|_{2} = \sqrt{\lambda_{\max}} \qquad \|\mathbf{A}\|_{\infty} = \max_{i} \|\tilde{\mathbf{a}}_{i}\|_{1}$$



