Linear Algebra & Random Processes

Random Variables and Probability Distributions

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Topics Covered & References

Topics

References

Random Variables

- A **random variable** (r.v.) is a real values function mapping the outcomes of a random experiment to a real number. Consider an experiment with sample space S, $\mathbf{x}(\bullet) : S \mapsto \mathbb{R}$; where, $\mathbf{x}$ is a random variable mapping S to a set of real numbers. The value of the random variable x for a specfic outcome of the sample space $\omega \in S$ is, $x = \mathbf{x}(\omega)$.
- In some cases a r.v. can be a simple numerical relabling of the orignal space, and in most cases they will act as some form of summary variable of the original sample space.
- The use of numerical values allows us to use standard tools of analysis to manipulate these variables for the purposes of understanding the nature of the random experiment.
- In a coin toss with sample space $S = \{H, T\}$, we could have a r.v. $\mathbf{x}(\omega) = \begin{cases} 1, & \omega = H \\ 0, & \omega = T \end{cases}$.
- If we toos 100 coins, then a r.v. could be the number of heads in the 100 tosses. What is the sample space represented by of this random variable?s

Randon Variables

- Our original experiment had a sample space S and a probability assignment function $P\left(\bullet\right) : S \mapsto [0, 1]$.
- When we define a r.v. $\mathbf{x} : S \mapsto \mathbb{R}$ for this experiment we have a new sample space $S_{\mathbf{x}}$ and a probability function $P_{\mathbf{x}}\left(\bullet\right) : \mathbb{R} \mapsto [0, 1]$; both can be derived from S and $P\left(\bullet\right)$.
- Let $S = \{o_1, o_2, \ldots o_n\}$, and $P\left(o_i\right) = p_i$. Let $S_{\mathbf{x}} = \{x_1, x_2, \ldots x_m\}$, where $x_i \in \mathbb{R}$.

$$P_{\mathbf{x}}\left(x_i\right) = P\left(\{o_j \,|\, \forall o_j \in S \text{ and } \mathbf{x}\left(o_j\right) = x_i\}\right)$$

- We toss three coins and observe what turns on the three coins.

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  2. $\mathbf{x} = \#H - \#T$. What is $S_{\mathbf{x}}$?

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  1. What is the sample space S?
  2. $\mathbf{x} = \#H - \#T$. What is $S_{\mathbf{x}}$?
  3. Prob. of H for each coin: p. What are $P(\bullet)$ and $P_{\mathbf{x}}(\bullet)$?

Random Variables

- We define events in terms of r.v.s, and determine their probability from S and $P(\bullet)$.

- We can condition a r.v. on an event or another r.v.

- The notion of independence can also be defined for an r.v. with respect to another r.v. or an event.

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  E.g. We toss three counts and define $\mathbf{x} = \#H - \#T$. What is $P_{\mathbf{x}}(x < 2)$?
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  E.g. We toss three counts and define $\mathbf{x} = \operatorname{sign}(\#H - \#T)$, $\mathbf{y} = \max\{\#H, \#T\}$, and $\mathbf{z} = \#H$. What is $P_{\mathbf{z}}(\bullet)$? What is $P_{\mathbf{x}|\mathbf{y}}(x|y = 2)$? What is $P_{\mathbf{x}|\mathbf{z}}(x|z = 2)$?

Probability Mass Function (PMF)

- Two types of r.v.s: **Discrete** and **Continuous**.
  - **Discrete**: Finite or countably infinite sample space.
  - **Continuous**: Uncountably infinite sample space.
- Probabilities associated with a discrete r.v. are fully characterized by the *probability mass function (PMF)*. Consider a r.v. $\mathbf{x}$. Let $f_{\mathbf{x}}$ by the PMF associated with $\mathbf{x}$ assigns probabilities to each element of $S_{\mathbf{x}}$.

$$f_{\mathbf{x}}(x) = P_{\mathbf{x}}(\mathbf{x} = x), \; x \in S_{\mathbf{x}}$$

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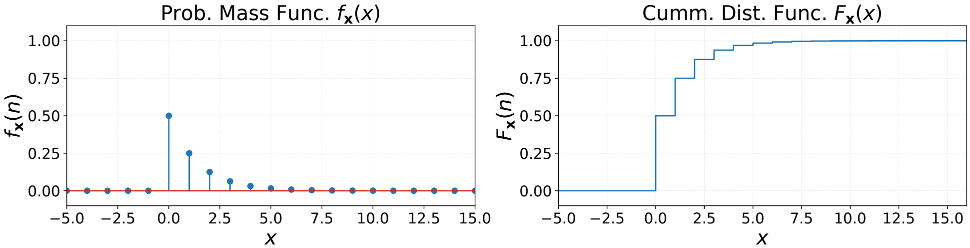
An urn contains 3 black balls, 4 white balls, and 5 red balls. We randomly select three balls from the urn with replacement, and count the number of red balls sampled $\mathbf{x}$. What is $p_{\mathbf{x}}(x)$?

Cummulative Distribution Function

- Cummulative distribution function (CDF) of $\mathbf{x}$.

$$F_{\mathbf{x}}(x) = P_{\mathbf{x}}(\mathbf{x} \leq x) = \sum_{\substack{y \leq x \\ y \in S_{\mathbf{x}}}} f_{\mathbf{x}}(y)$$

Consider a PMF $f_{\mathbf{x}}(x) = \begin{cases} \frac{1}{2}\left(\frac{1}{2}\right)^x & x \geq 0 \\ 0 & x < 0 \end{cases}, \quad x \in \mathbb{Z}$.



Bionimal Distribution

Bernoulli trials: Independent repeated trials with only two possible outcomes (success or failure), with fixed probabilities.
In n Bernoulli trials we are interested in the number of successes irrespective of their ordering.
The number of successes $\mathbf{x}$ is the **Binomial random variable**.

Sample space of $\mathbf{x}$: $S_{\mathbf{x}} =$

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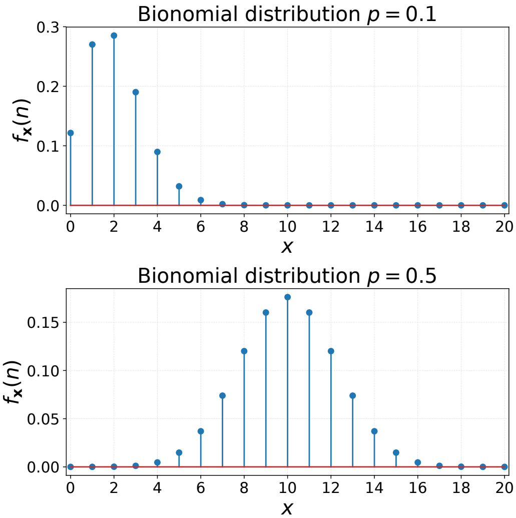

Geometric Distribution

Here again, we are dealing with Bernoulli trails, but instead of the number of successes, we are interested in the number of trials $\mathbf{x}$ required to obtain the first success.
We assume the probability of success on any trial is p.

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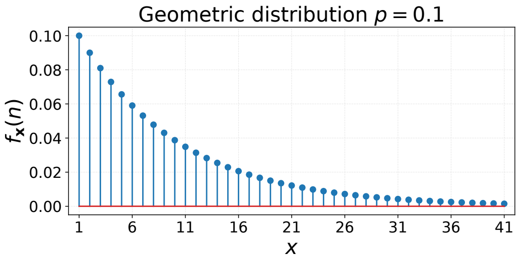

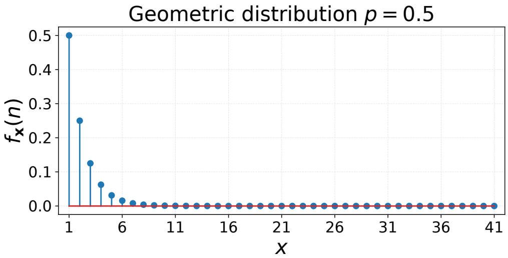

Poisson Distribution

We also often interested in the number of occurances of an event in a given interval of time or space. This count is a **Poisson** r.v., generated by a process with the following properties,

1. Only one event can occur at a time or location; simultaneous event are not allowed.
2. Occurrence of an event in a given period is independent occurrences in other non-overlapping periods.
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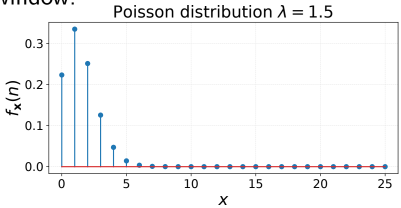
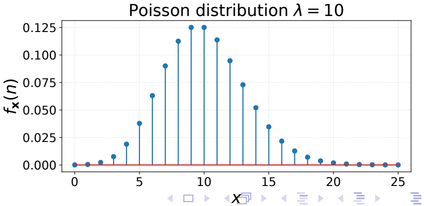
$$f_{\mathbf{x}}(x; t) = e^{-\lambda t} \frac{(\lambda t)^{-x}}{x!}$$

where, t is the fixed duration of interest, and λ is the average number of occurrences in a unit time window.





Continuous Random Variables

- We often encounter uncountable sample spaces, and we can associate these samples spaces with random variables that assume values in a continuous interval. E.g. Time taken to complete a task, blood glucose concentration, wait time for a doctor's appointment etc.
- Although in practice we only work with number of finite precision, working with a continuous random variable often simplifies analysis through the use of calculus.
- Consider a uncountably infinite sample space S, and a associated r.v. that maps to the entire real line $\mathbb{R}$, i.e. $\mathbf{x}(\bullet) : S \mapsto \mathbb{R}$.
- In this case, probabilities of events of S cannot be determined from probabilities to the elementary events.

Continuous Random Variables

- A simple way to assign probability to events in S is to start with events of the form $\mathbf{x} \leq x$. Probabilities of all other events of interest can be obtained from these probabilities $P(\mathbf{x} \leq x)$.
- We obtain the probability distribution function of a r.v. $\mathbf{x}$ by assinging probabilities to these events,
$$F_{\mathbf{x}}(x) \triangleq P(\mathbf{x} \leq x), \quad x \in (-\infty, \infty)$$
- The continuous analogue of the PMF in the discrete r.v. case is the probability density function (PDF) of the r.v. $\mathbf{x}$,
$$f_{\mathbf{x}}(x) \triangleq \frac{d}{dx} F_{\mathbf{x}}(x) \implies F_{\mathbf{x}}(x) = \int_{-\infty}^{x} f_{\mathbf{x}}(l)\, dl$$
- For small values of δx, $f_{\mathbf{x}}(x_0)\, \delta x$ is the probability $\mathbf{x}$ assuming values between x_0 and $x_0 + \delta x$.
$$P(x_0 \leq \mathbf{x} \leq x_0 + \delta x) = f_{\mathbf{x}}(x_0)\, \delta x$$

Properties of the prob. distribution function

- Prob. distribution functions of discrete r.v.s have step discontinuities, while that of continuous r.v.s are continuous.
- Discrete r.v.s can be represented using a prob. density function with impulse functions. This would be consistent with the step discontinuities in the prob. distribution function.

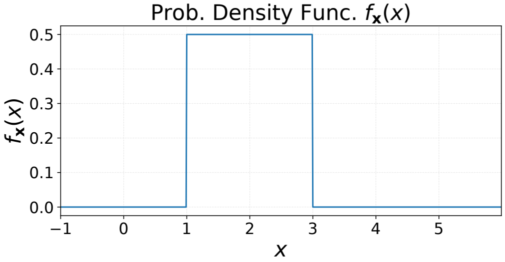
Properties of the prob. distribution function

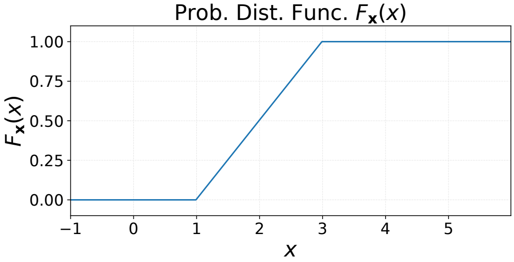
- $F_{\mathbf{x}}(-\infty) = 0$ and $F_{\mathbf{x}}(\infty) = 1$.
- If $x_1 < x_2$, then $F_{\mathbf{x}}(x_1) \leq F_{\mathbf{x}}(x_2)$.
- $P(\mathbf{x} > x) = 1 - F_{\mathbf{x}}(x)$
- $P(x_1 < \mathbf{x} \leq x_2) = F_{\mathbf{x}}(x_2) - F_{\mathbf{x}}(x_1)$
- $P(\mathbf{x} = x) = F_{\mathbf{x}}(x) - F_{\mathbf{x}}(x^-)$
- $P(x_1 \leq \mathbf{x} \leq x_2) = F_{\mathbf{x}}(x_2) - F_{\mathbf{x}}(x_1^-)$

Uniform Distribution

- This is encoutnered when sampling from a continuous region with all points in the region being equally likely.

$$f_{\mathbf{x}}(x; a, b) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}, \quad F_{\mathbf{x}}(x; a, b) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

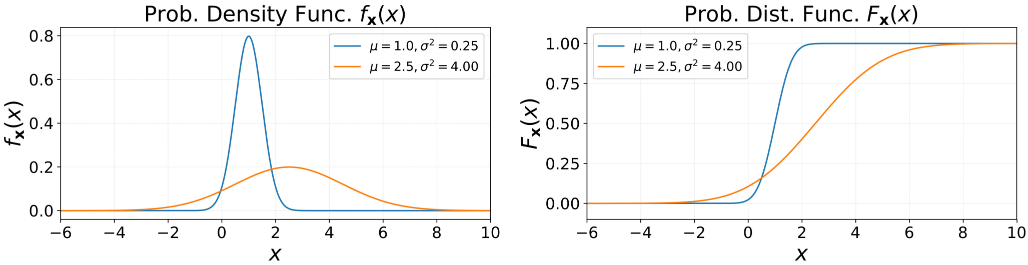




Gaussian Distribution

- Most common distribution encountered in probability and statitics.
- Central limit theorem states that the sum a large number of independent random variables tends to a Gaussian.

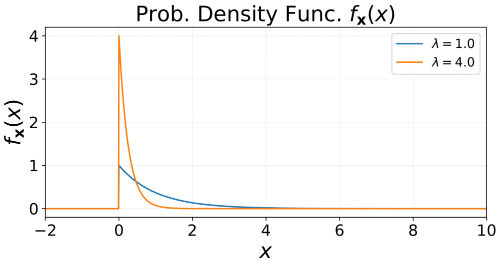
$$f_{\mathbf{x}}\left(x; \mu, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \mathcal{N}\left(\mu, \sigma^2\right), \quad F_{\mathbf{x}}\left(x; a, b\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du$$

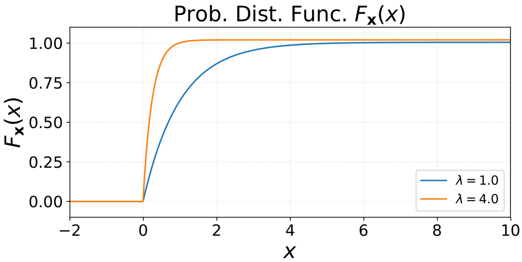


Exponential Distribution

- This is encountered when dealing with problems of waiting times between events, where the occurance of an event in two non-overlapping time windows is independent.

$$f_{\mathbf{x}}(x;\lambda) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}, \quad F_{\mathbf{x}}(x;\lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$





Conditional Probability Distributions and Densities

We can extend the idea of conditional probability can be extended to probability distributions. Consider a random variable $\mathbf{x}$, and some event A that has occured. We can determine the **conditional probability distribution** as the following,

$$F_{\mathbf{x}}\left(x|A\right) \triangleq P\left(\mathbf{x} \leq x|A\right) = \frac{P\left(\mathbf{x} \leq x, A\right)}{P\left(A\right)}, \; P\left(A\right) = 0$$

where, $P\left(\mathbf{x} \leq x, A\right)$ is the probability of the intersection of the events $\mathbf{x} \leq x$ and A.
We can similarly define the **conditional probability density function**,

$$f_{\mathbf{x}}\left(x|A\right) \triangleq \frac{dF_{\mathbf{x}}\left(x\right)}{dx} = P\left(\mathbf{x} \leq x|A\right) = \frac{P\left(\mathbf{x} \leq x, A\right)}{P\left(A\right)}, \; P\left(A\right) = 0$$

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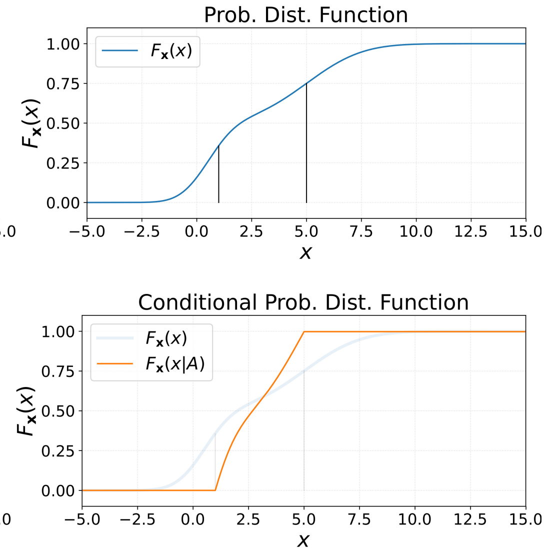

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A random binary signal $\mathbf{x} \in \{0, 1\}$ is transmitted through a additive Gaussian noisy channel whose noise properties depend on the voltage being transmitted $(P(\mathbf{x} = 1) = p)$. The received signal amplitude at each ppint in time is a random varible $\mathbf{r}$, $\begin{cases} \mathcal{N}(0, \sigma_1^2), & x = 0V \\ \mathcal{N}(1, \sigma_2^2), & x = 1V \end{cases}$. Find the expression for $f_{\mathbf{r}}(r)$.

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- If B is the event $x_1 < \mathbf{x} \le x_2$, then we have,

$$P(A \,|\, x_1 < \mathbf{x} \le x_2) = \frac{F_{\mathbf{x}}(x_2 \,|\, A) - F_{\mathbf{x}}(x_1 \,|\, A)}{F_{\mathbf{x}}(x_2) - F_{\mathbf{x}}(x_1)} P(A)$$

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- Can we find out $P(A \,|\, \mathbf{x} = x)$?
$$P(A \,|\, \mathbf{x} = x) = \lim_{\Delta x \to 0} P(A \,|\, x < \mathbf{x} \leq x + \Delta x) = \frac{f_{\mathbf{x}}(x \,|\, A)}{f_{\mathbf{x}}(x)} P(A)$$

Baye's Theorem

$$P(\mathbf{x}=1 \,|\, \mathbf{r}=0.2) = ?$$
$$P(\mathbf{x}=1 \,|\, \mathbf{r}=0.4) = ?$$
$$P(\mathbf{x}=1 \,|\, \mathbf{r}=0.6) = ?$$